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Linear analysis of forced magnetic reconnection due to a boundary perturbation
強制磁気再結合の線形解析

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Abstract

The forced magnetic reconnection due to the boundary perturbation is investigated analytically by use of the boundary layer theory. A new reconnected flux is derived with the exact asymptotic matching and a time dependent imposition of the boundary perturbation. By virtue of the exact matching, the effect of the inertia of the plasma in the inner layer is correctly included. At the initial evolution, the magnetic field lines reconnect on the time scale which includes the time scale of the imposition of the boundary perturbation and it can be faster than the Sweet-Parker time scale. The local current is induced on the resonant surface to suppress the growth of magnetic islands at the initial evolution. Moreover the equation for the time evolution of the reconnected flux is proposed in terms of an integral equation.

1 Introduction

In plasma confinement, there are two kinds of the magnetic reconnections: free reconnection and forced reconnection. The free reconnection is the spontaneous instability such as the tearing mode[1]. Although a magnetic equilibrium is stable for the free reconnection, an externally imposed boundary perturbation forces to give rise to the magnetic reconnection on the resonant surface; it is called forced reconnection.[2] The energy source of the perturbation of the forced reconnection is the boundary perturbation, while that of the free reconnection is the equilibrium magnetic field.

The forced reconnection occurs in the magnetic island formation due to the resonant magnetic field error[2] and in the seed islands formation for the neo-classical tearing mode due to the geometrically coupled perturbation[3] in the plasma confinement such as tokamaks. The error field is the small deviation from axial symmetry of the magnetic field lines and it perturbs the plasma boundary. In the later case, as a model, the boundary perturbation expresses the toroidal coupling with a magnetic signal produced by another MHD instability.

The response of the plasma to the applied boundary perturbation is described by the simple model[2] which is fruitful for the analytical study. In this model, the perturbation is caused by a deformation of the plasma boundary. The ideal MHD equations for this deformation of the boundary yields two equilibriums with different topologies. One magnetic equilibrium has the same topology as the original equilibrium with a local current sheet on the resonant surface. The other has the different topology with magnetic islands on the resonant surface without the current sheet. The former is called equilibrium (I), and the latter is called equilibrium (II).[2] The existence of the equilibrium (II) implies that the boundary perturbation can change the topology of the magnetic field lines and give rise to the forced magnetic reconnection to construct the magnetic islands on the resonant surface.

The time evolution of the forced reconnection process is investigated by use of the boundary layer theory.[2, 4, 5] The analysis of linear evolution is important, since it affects to the subsequent nonlinear evolution. In the previous linear analysis, the time scale of the initial evolution of the forced reconnection is believed to be Sweet-Parker time scale. We revealed that this time scale stems from the using of the matching condition which is valid only in the constant-ψ approximation; the effect of the inertia of the plasma in the inner layer is neglected in this matching condition. However it is important to include the effect of the inertia as well as the resistivity in the analysis of the forced reconnection.

In this paper we correct the analysis in the previous works[2, 4, 5] in order to obtain the

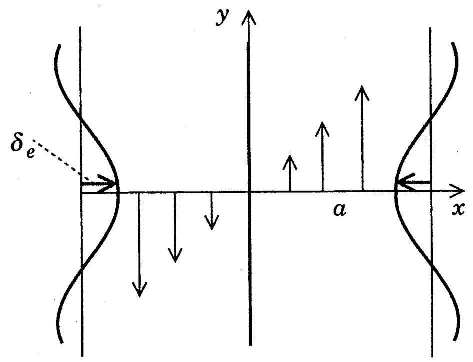

Figure 1: Coordinate system for the slab of incmpressible plasma

exact linear evolution of the forced reconnection. First, we adopt the exact matching condition and use the exact solution for the inner layer equation to take into account the effect of the inertia in the inner layer, correctly. Second, in the previous works, although the imposition of the boundary perturbation is assumed to be much slower than the Alfven time scale, the plasma boundary is deformed suddenly except in Ref.[3]. Thus we correct this point and consider the time dependent imposition of the boundary perturbation so that the outer region obeys the ideal MHD equilibrium equations.

The paper is organized as follows. We describe the model and the method of analysis in section 2. In section 3, a new Laplace transformed reconnected flux based on the exact matching condition is presented. With this condition, the initial evolution of the forced reconnection is calculated in section 4. In section 5, the time evolution equation of the reconnected flux is introduced. Finally section 6 is devoted to the summary and discussion.

2 Model and Equations

In this section we shall present the basic equations and recall some fundamental properties of the boundary layer theory for the forced magnetic reconnection. In the boundary layer theory, we separate the entire plasma into two regions. One is the outer region, where the plasma is quasi-static and governed by the ideal MHD equations. The other is the vicinity of the resonant surface, where the inertia and resistivity of the plasma are important; it is called inner layer. Asymptotic matching of the two regions yields equations for the time evolution of magnetic islands.

2.1 Outer region

In order to investigate the process of the forced reconnection, we consider the response of the plasma to the applied boundary perturbation on the equilibrium. We shall consider a slab of incompressible plasma bounded by two parallel perfectly conducting walls. The magnetic field is represented as $\boldsymbol{B} = B_T \boldsymbol{e}_z + \boldsymbol{e}_z \times \nabla\psi$, where B_T stands for the uniform toroidal field and ψ is a magnetic potential. We take the coordinate with the xy-plane normal to the toroidal field B_T and the y-axis parallel to the wall and the x-axis normal to it. The magnetic equilibrium is governed by the ideal MHD equations,

$$\nabla \times (\boldsymbol{j} \times \boldsymbol{B}) = 0, \tag{1}$$

where $\boldsymbol{j} = \nabla \times \boldsymbol{B}/4\pi$ is the current density.

In the absence of the boundary perturbation, we have the static equilibrium $\psi = \psi_0(x)$ subjected to the boundary conditions $\psi_0(x = \pm a) = const.$ where a is the half of the plasma width. This equilibrium is assumed to have the resonant surface at the center of plasma, $x = 0$, and supposed to be stable for the usual tearing mode, such as $\psi_0 = B_0 x^2/2a$ for the Taylor's model[2].

Here we consider the imposition of the boundary perturbation to the initial static equilibrium. The externally imposed boundary perturbation is described by means of the deformation of the plasma boundary as

$$\psi(x = \pm(a - \delta(t/\tau_e)\cos ky)) = const.,$$

where k, $\delta_e(t/\tau_e)$ and τ_e are wave number, time dependent amplitude and imposition time scale of the boundary perturbation, respectively. The boundary perturbation is very weak, $\delta(t/\tau_e) \ll a$, such as the error field. We assume that imposition of the boundary perturbation is much slower than the Alfven time scale $\tau_A = a/v_A$ so that the outer region is always in equilibrium and obeys the ideal MHD equations, and much faster than any resistive time $\tau_R = 4\pi a^2/\eta$,

$$\tau_A \ll \tau_e \ll \tau_R,$$

where $v_A = B_0/(4\pi\rho)^{1/2}$ is the Alfven speed, and η and ρ are the normalized resistivity and the density of the plasma respectively. However, in the previous works, the outer region is assumed to obey the ideal MHD equations and the sudden imposition, $\delta(t/\tau_e) = \delta\theta(t)$, is considered, where θ is the Heaviside function; these contradict each other. Therefore we should consider the slowly varying imposition of the boundary perturbation.

The magnetic equilibrium perturbed by the boundary deformation is written as

$$\psi(x,t) = \psi_0(x) + \psi_1(x,t)\cos ky, \qquad (2)$$

where $\psi_1(x,t)$ denotes the perturbed part due to the boundary perturbation. Since the boundary perturbation is imposed on the time scale much slower than the Alfven time scale, the plasma is quasi-static and obeys the ideal MHD equations (1) except the vicinity of the resonant surface, where $x = 0$. The ideal MHD equations (1) for the perturbation $\psi_1(x,t)$ is

$$B_{0y}(x)\left\{\frac{\partial^2\psi_1(x,t)}{\partial^2 x} - k^2\psi_1(x,t)\right\} = 0, \qquad (3)$$

with the boundary condition

$$\psi_1(\pm a) = \delta(t/\tau_e)B_{0y}(a) \equiv \psi_e(t/\tau_e),$$

where $B_{0y}(x) = d\psi_0(x)/dx$. The solution to this equation, $\psi_1(x,t)$, should be a even function for x, since the equation (3) and the boundary condition are unchanged for $x \to -x$. Thus the quasi-equilibrium state as the solution to the equation (3) can be written as

$$\psi_1(x,t) = \psi_1(0,t)f(x) + \psi_e(t/\tau_e)g(x), \qquad (4)$$

where $f(x)$ stands for the eigenfunction for the usual tearing mode subjected to the boundary conditions $f(0) = 1$ and $f(\pm a) = 0$ and $g(x)$ is the response to the imposed boundary perturbation which satisfies the boundary conditions $g(0) = 0$ and $g(\pm a) = 1$.[3, 6] These functions satisfies the ideal MHD equation (3), respectively. The first term corresponds to the reconnected flux and the second term corresponds to the shielded flux for the cylindrical geometry.[6, 8] The time dependent coefficient $\psi_1(0,t)$ is the magnetic potential on the resonant surface and expresses the amount of reconnected flux at the resonant surface; here after we call it reconnected flux. Since the imposition function, $\psi_e(t/\tau_e)$, is a given function, the time evolution of the forced reconnection is described only by the reconnected flux $\psi_1(0,t)$[2].

In order to determine the reconnected flux, $\psi_1(0,t)$, we consider the initial value problem by applying the Laplace transform

$$\tilde{f}(x,s) = \int_0^\infty f(x,t)e^{-st}dt,$$

to the equation (4). The initial condition for the perturbation is $\psi_1(x,0) = 0$, since there is no deformation of the boundary $\psi_e(0) = 0$ at $t = 0$. Demanding that the Laplace-transformed outer solution matches asymptotically to the inner layer solution, we will have the matching condition in section 3. The Laplace-transformed outer solution can be expanded asymptotically as

$$\tilde{\psi}_1(x,s) \approx \tilde{\psi}_1(0,s)\left\{1 + \frac{\Delta'_{outer}}{2}x + \cdots\right\}, \qquad (5)$$

as $x \to +0$ where

$$\begin{aligned}\Delta'_{outer}(s) &\equiv \frac{1}{\tilde{\psi}_1(0,s)}\left[\frac{d\tilde{\psi}_1(x,s)}{dx}\right]_{-0}^{+0} \\ &= \Delta'_0 + \Delta'_s\frac{\tilde{\psi}_e(s)}{\tilde{\psi}_1(0,s)}, \qquad (6)\end{aligned}$$

where

$$\Delta'_0 = \left[\frac{df(x)}{dx}\right]_{-0}^{+0}, \qquad \Delta'_s = \left[\frac{dg(x)}{dx}\right]_{-0}^{+0},$$

are the stability parameter for the usual tearing mode in the absence of the boundary perturbation and the deviation from it due to the boundary perturbation. Since the initial equilibrium is supposed to be stable, Δ'_0 is negative.

For instance, in the Taylor's model, $f(x) = G(x) - G(a)F(x)/F(a)$, $g(x) = F(x)/F(a)$, $\Delta'_0 = -2kG(a)/F(a)$ and $\Delta'_s = 2k/F(a)$ where $F(x) = |\sinh kx|$, $G(x) = \cosh kx$. The (I) sate is realized when $\psi_1(0,t) = 0$. On the other hand $\psi_1(0,t) = B_0\delta/\cosh ka$ can be regarded as the full reconnected state corresponding to the (II) state which has the magnetic islands with the width, $2\sqrt{2a\psi_1(0)/B_0}$.

2.2 Inner Layer

As seen in the previous subsection, the time development of the forced reconnection as the quasi-equilibrium state is determined only by the reconnected flux $\psi_1(0,t)$. However the ideal MHD equation cannot determine the time evolution of it. In order to obtain the reconnected flux, we should investigate the dynamics in the vicinity of the resonant surface, i.e. the inner layer, where the effect of the inertia and resistivity should be included. The inner layer obeys the reduced MHD equations,

$$\rho\left(\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\nabla\right)\nabla^2\varphi=\boldsymbol{B}\cdot\nabla j_z, \tag{7}$$

$$\frac{\partial\psi}{\partial t}+\boldsymbol{B}\cdot\nabla\varphi=\frac{\eta}{4\pi}\nabla^2\psi, \tag{8}$$

where $j_z=\nabla^2\psi/4\pi$ and $\boldsymbol{v}=\boldsymbol{e}_z\times\nabla\varphi$ indicate z-component of the current density and the velocity of the plasma respectively, and $\varphi=\varphi_1(x)\sin ky$ is a static potential or stream function. Since the deformation of the boundary is very small, the perturbation ψ_1 is small at the initial evolution. Thus the perturbed quantities obey the linearized reduced MHD equations. We consider the initial value problem and apply the Laplace transform to the linearized reduced MHD equations with the initial condition $\psi_1(x,0)=\varphi_1(x,0)=0$ and stretch the x-axis in the vicinity of the resonant surface with the ratio εa, where $\varepsilon^4=s\tau_A^2/(4(ka)^2\tau_R)$, then we have the equations in the inner layer,

$$4\varepsilon\Omega\frac{d^2U}{d\theta^2}=\theta\frac{d^2\Psi}{d\theta^2}, \tag{9}$$

$$\frac{d^2\Psi}{d\theta^2}=\varepsilon\Omega(4\Psi+\theta U), \tag{10}$$

where $U=-4\varepsilon k^2\tilde{\varphi}_1/s$ and $\Psi=k\tilde{\psi}_1/B_0$ are the normalized stream function and magnetic potential respectively, and $\theta=x/\varepsilon a$ is the stretched coordinate and $\Omega=\varepsilon\tau_R s/4$. Then it follows from the eqs. (9) and (10) that the inner layer equation as[9]

$$\frac{d^2\chi}{d\theta^2}-\frac{2}{\theta}\frac{d\chi}{d\theta}-(4\varepsilon\Omega+\frac{\theta^2}{4})\chi=-\frac{\chi_\infty}{4}\theta^2, \tag{11}$$

where

$$\chi\equiv4\varepsilon\Omega\frac{dU}{d\theta}+\chi_\infty=\theta^2\frac{d}{d\theta}(\frac{\Psi}{\theta}). \tag{12}$$

This equation corresponds to the equation of χ in Ref. [9] by rewriting the variables $\theta\to\sqrt{2}\hat{x}$ and $2\varepsilon\Omega\to\hat{\lambda}^{3/2}/4$.

Following Ara et. al.[9] we obtain the solution to the inner layer equation (11) without any approximation such as the analysis in the previous works.[2, 4, 5] The solution has the form

$$\begin{aligned}\chi &= \chi_\infty-\chi_\infty\frac{2\varepsilon\Omega}{\sqrt{2}}\int_0^1 y^{2\varepsilon\Omega-5/4}\sqrt{1+y}\\ &\quad\times\exp\left(\frac{-\theta^2}{4}\frac{1-y}{1+y}\right)dy.\end{aligned} \tag{13}$$

Since the solution at the outer region has the symmetry $\psi_1(-x)=\psi_1(x)$, U and Ψ should be odd and even functions for θ, respectively. Integrating the equation (12) to satisfy these parity gives the solution for positive θ as

$$U=\frac{1}{4\varepsilon\Omega}\int_0^\theta(\chi-\chi_\infty)d\theta,$$

$$\begin{aligned}\Psi(\theta) &= -\chi+\theta\int_0^\theta\frac{1}{\theta}\frac{d\chi}{d\theta}d\theta\\ &= -\chi_\infty+\chi_\infty\frac{2\varepsilon\Omega}{\sqrt{2}}\int_0^1 y^{2\varepsilon\Omega-5/4}\sqrt{1+y}\\ &\quad\times\exp\left(\frac{-\theta^2}{4}\frac{1-y}{1+y}\right)dy \quad (14)\\ &\quad+\chi_\infty\theta\frac{\sqrt{\pi}2\varepsilon\Omega}{2\sqrt{2}}\int_0^1 y^{2\varepsilon\Omega-5/4}\sqrt{1-y}\\ &\quad\times\mathrm{erf}\left(\frac{\theta}{2}\frac{\sqrt{1-y}}{\sqrt{1+y}}\right)dy, \quad (15)\end{aligned}$$

where erf indicates the error function and the normalization factor χ_∞ is related to the magnetic potential at the neutral surface in the inner layer, $\Psi(0)$, as

$$\chi_\infty=\frac{\Psi(0)}{\frac{2\varepsilon\Omega}{2\varepsilon\Omega-1/4}F(1,-1/2,2\varepsilon\Omega+3/4,1/2)-1}, \tag{16}$$

where F is the Gauss's Hypergeometric function.

The asymptotic expansion of Ψ can be written as

$$\Psi(\theta)\approx-\chi_\infty\left\{1-\frac{2\varepsilon\Omega\pi}{4\sqrt{2}}\frac{\Gamma(2\varepsilon\Omega-1/4)}{\Gamma(2\varepsilon\Omega+5/4)}\theta+\cdots\right\} \tag{17}$$

as $\theta\to+\infty$ where Γ is the gamma function and

$$2\varepsilon\Omega=\frac{(s\tau_A^{2/3}\tau_R^{1/3})^{3/2}}{4ka},\quad \Omega=\frac{(s\tau_A^{2/5}\tau_R^{3/5})^{5/4}}{4\sqrt{2}ka}.$$

3 Reconnected flux with exact asymptotic matching

Demanding that the solution for the inner layer equation matches asymptotically with the solution at the outer region yields the matching conditions. The matching conditions give the Laplace-transformed reconnected flux which determines the time evolution of the magnetic islands due to the forced reconnection.

Here we adopt the exact matching condition, while the matching condition adopted in the previous works is available only in the constant-ψ approximation. In order to be clarify this point, we rewrite the asymptotic expansion of Ψ, (17), as

$$\Psi(\theta) \approx \Psi_\infty \left\{1 + \frac{\Delta'_{inner}}{2}x + \cdots\right\} \tag{18}$$

for $\theta \to +\infty$ where

$$\Psi_\infty = -\chi_\infty, \tag{19}$$

$$\begin{aligned}\Delta'_{inner}(s) &= \frac{1}{\varepsilon a}\frac{1}{\Psi_\infty}\left[\frac{d\Psi}{d\theta}\right]_{-\infty}^{\infty} \\ &= \frac{-\pi\Omega}{\sqrt{2}a}\frac{\Gamma(2\varepsilon\Omega - 1/4)}{\Gamma(2\varepsilon\Omega + 5/4)}.\end{aligned} \tag{20}$$

In the previous analysis,[2, 4, 5] $[d\Psi/d\theta]_{-\infty}^{+\infty}$ is divided by $\Psi(0)$ instead of Ψ_∞ in the equation (20). That is valid only in the constant-ψ approximation which is neglect the effect of the inertia of the plasma in the inner layer. The effect of the inertia makes $\Psi(0)$ deviate from $\Psi_\infty = -\chi_\infty$ as shown in the equation (16).

Asymptotic matching of (5) and (18) yields the exact matching conditions which include the effect of the inertia of the plasma in the inner layer, correctly, as

$$\tilde{\psi}_1(0,s) = \frac{B_0}{k}\Psi_\infty, \tag{21}$$

$$\Delta'_{outer} = \Delta'_{inner}. \tag{22}$$

Combining the matching conditions (21) and (22), and the equation (6), we have the exact Laplace-transformed reconnected flux

$$\tilde{\psi}_1(0,s) = \frac{\Delta'_s\tilde{\psi}_e(s)}{\Delta'_{inner}(s) - \Delta'_0}, \tag{23}$$

based on the boundary layer analysis of the linearized reduced MHD equations without any approximations. In the absence of the boundary perturbation, $\psi_e = 0$, the initial value problem reduces to the eigenvalue problem and the equation (23) gives the dispersion relation of the general resistive modes, $\Delta'_{inner}(s) - \Delta'_0 = 0$[9, 10, 12].

Exactly saying, there are two reconnected fluxes. One is the reconnected flux at $x = 0$, $\tilde{\psi}_1(0,s)$, which represents the changing of the equilibrium with the global deformation of the magnetic field lines as seen in the equation (4). The other is the reconnected flux at the origin of the stretched coordinate $\theta = 0$, $\Psi(0)$, which represents the reconnected flux at the neutral surface in the inner layer and is called inner-layer reconnected flux in this paper. The inertia of the plasma affects on the inner-layer reconnected flux, $\Psi(0)$, to be different from the reconnected flux, $\tilde{\psi}_1(0,s)$. Although reconnected flux $\psi_1(0,t)$ represents the global deformation of the magnetic field lines by the boundary perturbation, it's increase expresses not only the deformation due to the reconnection, but also the ideal deformation by the boundary perturbation.

Combining the equations (16), (19) and the matching condition (21) we have the Laplace-transformed inner-layer reconnected flux as,

$$\begin{aligned}\Psi(0) = \frac{k}{B_0}&\left\{1 - \frac{2\varepsilon\Omega}{2\varepsilon\Omega - 1/4}\right. \\ &\left.\times F(1, -1/2, 2\varepsilon\Omega + 3/4, 1/2)\right\}\tilde{\psi}_1(0,s)\end{aligned} \tag{24}$$

This difference in the equation (24) corresponds to the one between reconnection rate R and $q\hat{c}$ in equation (18) in Ref. [11]. For the low growth rate limit, $s \to 0$, the equation (24) is reduced to

$$\Psi(0) = \frac{k}{B_0}\tilde{\psi}_1(0,s), \tag{25}$$

to validate the constant-ψ matching condition $\Delta'_{outer} = [d\Psi/d\theta]_{-\infty}^{\infty}/(\varepsilon a\Psi(0))$ [2]. This constant-ψ matching condition leads to the initial evolution with the Sweet-Parker time scale.

4 Initial evolution

We calculate the initial evolution of the reconnected flux by use of the theorem that the Taylor series expansion of a function $f(t)$ at $t = 0$

corresponds to the asymptotic power expansion of the Laplace-transformed function $\tilde{f}(s)$ for $s \to \infty$.

Here we consider the imposition of the boundary perturbation. As mentioned above the sudden imposition of the boundary perturbation in the previous works contradicts to the assumption that the outer region is the quasi-static ideal equilibrium. In fact this imposition leads to the unphysical result. Therefore we have to adopt the time dependent imposition. The imposition function is assumed to be even for t, for simplicity, then it can be expanded as

$$\psi_e(t/\tau_e) \approx \frac{\psi_e''(0)}{2!}\frac{t^2}{\tau_e^2}+\frac{\psi_e''''(0)}{4!}\frac{t^4}{\tau_e^4}+\frac{\psi_e''''''(0)}{6!}\frac{t^6}{\tau_e^6}+\cdots$$

since $\psi_e(0) = 0$.

With this time varying imposition, the Laplace-transformed reconnected flux (23) can be asymptotically expanded in s as

$$\begin{aligned}\tilde{\psi}_1(0,s) \approx\ & -\frac{\Delta_s'}{\Delta_0'}\left\{\frac{\psi_e''(0)}{\tau_e^2 s^3}+\frac{\psi_e''(0)}{\tau_\alpha\tau_e^2 s^4}\right.\\ & +\left(\frac{\psi_e''(0)}{\tau_e^2\tau_\alpha^2}+\frac{\psi_e''''(0)}{\tau_e^4}\right)\frac{1}{s^5}\\ & +\left(\frac{\psi_e''(0)}{\tau_e^2\tau_\alpha^2}+\frac{\psi_e''''(0)}{\tau_e^4}\right)\frac{1}{\tau_\alpha s^6}\\ & +\left(\frac{\psi_e''''''(0)}{\tau_e^6}+\frac{\psi_e''(0)}{\tau_e^2\tau_\alpha^4}+\frac{\psi_e''''(0)}{\tau_e^4\tau_\alpha^2}\right.\\ & \left.\left.+\frac{4^2\psi_e''(0)}{\tau_e^2\tau_\alpha\tau_c^3}\right)\frac{1}{s^7}+\cdots\right\} \qquad (26)\end{aligned}$$

for $s \to \infty$, where

$$\tau_\alpha = \frac{-\Delta_0'}{\pi k}\tau_A, \qquad \tau_c = \frac{\tau_A^{2/3}\tau_R^{1/3}}{(ka)^{2/3}},$$

denote the ideal time scale and the typical time scale of the inner layer, respectively. The inversion of Laplace transform of this equation gives the Taylor expansion of the reconnected flux as

$$\begin{aligned}\psi_1(0,t) =\ & -\frac{\Delta_s'}{\Delta_0'}\left\{\frac{\psi_e''(0)}{\tau_e^2}\frac{t^2}{2!}+\frac{\psi_e''(0)}{\tau_\alpha\tau_e^2}\frac{t^3}{3!}\right.\\ & +\left(\frac{\psi_e''(0)}{\tau_e^2\tau_\alpha^2}+\frac{\psi_e''''(0)}{\tau_e^4}\right)\frac{t^4}{4!}\\ & +\left(\frac{\psi_e''(0)}{\tau_e^2\tau_\alpha^2}+\frac{\psi_e''''(0)}{\tau_e^4}\right)\frac{t^5}{\tau_\alpha 5!}\\ & +\left(\frac{\psi_e''''''(0)}{\tau_e^6}+\frac{\psi_e''(0)}{\tau_e^2\tau_\alpha^4}+\frac{\psi_e''''(0)}{\tau_e^4\tau_\alpha^2}\right.\\ & \left.\left.+\frac{4^2\psi_e''(0)}{\tau_e^2\tau_\alpha\tau_c^3}\right)\frac{t^6}{6!}+\cdots\right\} \qquad (27)\end{aligned}$$

This reconnected flux vanishes at $t = 0$ to satisfy the initial condition. Here we consider the time scale of the reconnection process. The first term is dominant at the initial evolution, therefore the reconnection occurs with the boundary perturbation imposition time scale, τ_e which can be faster than the Sweet-Parker time scale. Hence it appears that the exact matching leads to the different time scale of the initial evolution from the Sweet-Parker time scale in the previous investigations. The each term in the Taylor series (27) consists of t/τ_e, t/τ_A and $t/\tau_A^{2/3}\tau_R^{1/3}$. and the resistive time scale in $\tau_c \propto \tau_A^{2/3}\tau_R^{1/3}$ appears at higher than the 5th order. It appears that the stability parameter for the tearing mode Δ_0' which is included in τ_α is important. The reconnected flux with large Δ_0' increased more slowly than that with small Δ_0'.

When the boundary perturbation is imposed, the local current is induced at the resonant surface. It is represented by the total current in the inner layer and is equivalent to the difference of the y component of the magnetic field at the resonant surface, $x = 0$,

$$\begin{aligned}\Delta B_y(t) \equiv\ & \left[\frac{\partial\psi_1(x,t)}{\partial x}\right]_{-0}^{+0}\\ =\ & \Delta_0'\psi_1(0,t)+\Delta_s'\psi_e(t). \qquad (28)\end{aligned}$$

This equation implies that the total current decays with the increase of the reconnected flux $\psi_1(0,t)$, and increases with the imposition function $\psi_e(t)$, since $\Delta_0' < 0$ for the stable equilibrium and $\Delta_s' > 0$. Substituting the equation (27) into (28) gives the initial evolution of the total current in the inner layer as

$$\begin{aligned}\Delta B_y(t) =\ & -\Delta_s'\left\{\frac{\psi_e''(0)}{3!}\frac{t^3}{\tau_\alpha\tau_e^2}+\frac{\psi_e''(0)}{\tau_\alpha^2\tau_e^2}\frac{t^4}{4!}\right.\\ & +\left(\frac{\psi_e''(0)}{\tau_e^2\tau_\alpha^2}+\frac{\psi_e''''(0)}{\tau_e^4}\right)\frac{t^5}{\tau_\alpha 5!}\\ & +\left(\frac{\psi_e''(0)}{\tau_e^2\tau_\alpha^4}+\frac{\psi_e''''(0)}{\tau_e^4\tau_\alpha^2}+\frac{4^2\psi_e''(0)}{\tau_e^2\tau_\alpha\tau_c^3}\right)\frac{t^6}{6!}\\ & \left.+\cdots\right\} \qquad (29)\end{aligned}$$

It grows with the negative sign, thus the stability parameter $\Delta' = \Delta B_y/\psi_1(0,t)$ is negative in the initial evolution and $\Delta' = 0$ at $t = 0$, while it is claimed that $\Delta' \to \infty$ at $t = 0$ in the previous works. The negative sign of the total current in the inner layer implies that the local current is induced on the resonant surface to suppress the growth of magnetic islands. This negative growth stems from the fact that the initial static equilibrium is stable, $\Delta'_0 < 0$.

As mentioned above there are two reconnected fluxes. The reconnected flux derived above is the one at the resonant surface for the outer variable $x = 0$. However the limit $x \to 0$ of the outer variable corresponds to the limit $\theta \to \infty$ of the inner variable as shown in the matching conditions. The magnetic and dynamic structures in the inner layer makes the reconnected flux at $x = 0$ to be different from the one at $\theta = 0$. Therefore the exact reconnected flux in the inner layer should be defined at $\theta = 0$: inner-layer reconnected flux. Although, in the constant-ψ approximation, these reconnected flux have the same value as shown in the equation (25), in the forced reconnection process the effect of the inertia makes these to be different values as shown in the equation (24). The inner-layer reconnected flux is defined as

$$\psi_{inner}(\theta = 0, t) \equiv L^{-1}[B_0\Psi(0)/k].$$

The inverse Laplace-transformation of the asymptotic expansion of the equation (24) with the equation (26) yields the Taylor expansion of the inner-layer reconnected flux as

$$\begin{aligned}\psi_{inner}(0,t) \;=\; & \frac{\Delta'_s}{\Delta'_0}\left\{\frac{2\psi''_e(0)}{\tau_e^2\tau_c^3}\frac{t^5}{5!} + \frac{2\psi''_e(0)}{\tau_\alpha\tau_e^2\tau_c^3}\frac{t^6}{6!}\right. \\ & + \frac{t^7}{7!\tau_e^2\tau_c^3}\left(\frac{\psi''_e(0)}{\tau_\alpha^2} + \frac{\psi''''_e(0)}{\tau_e^2}\right) \\ & + \left(\frac{2\psi''_e(0)}{\tau_\alpha^3\tau_e^2\tau_c^3} + \frac{2\psi''''_e(0)}{\tau_e^4\tau_\alpha\tau_c^3}\right. \\ & \left.\left. - \frac{4\psi''_e(0)}{\tau_e^2\tau_c^6}\right)\frac{t^8}{8!} + \cdots\right\} \qquad (30)\end{aligned}$$

It evolves with the time scale more close to the inner-layer time scale $\tau_c \propto \tau_A^{2/3}\tau_R^{1/3}$ than the imposition time scale which is dominant in the initial evolution of the reconnected flux $\psi_1(0,t)$. Therefore the inner-layer reconnected flux can also increase faster than the Sweet-Parker time scale.

5 Time evolution equation for reconnected flux

In the preceding section we obtained the initial evolution. In this section we propose the new method to determine the time evolution of the reconnected flux which can describe the evolution subsequent to the initial evolution. The Laplace-transformed equation of the reconnected flux (23) can be rewritten as

$$\tilde{\psi}_1(0,s) - \frac{\Delta'_{inner}(s)}{\Delta'_0}\tilde{\psi}_1(0,s) = \frac{-\Delta'_s}{\Delta'_0}\tilde{\psi}_e(s).$$

The inversion of Laplace transform of this equation gives the following inhomogeneous second kind Volterra equation as

$$\psi_1(0,t) - \frac{1}{\Delta'_0}\int_0^t \psi_1(0,\tau)G(t-\tau)d\tau = \frac{-\Delta'_s}{\Delta'_0}\psi_e(t), \qquad (31)$$

where the kernel $G(t)$ is the inverse of the Laplace transform of $\Delta'_{inner}(s)$ and written as

$$\begin{aligned}G(t) \;=\; & \frac{-4k}{3\tau_A}\left\{\frac{\sqrt{\pi}}{2}\exp(\frac{t}{\tau_c}) + \sum_{n=1}^{\infty}\frac{\sqrt{n-1/4}}{n!}\right. \\ & \left.\times\Gamma(n-1/2)\exp(\frac{-t}{2\tau_n})\sin(\frac{\sqrt{3}}{2}\frac{t}{\tau_n})\right\} \\ & + \frac{k}{3\pi\tau_A}\int_0^\infty \sqrt{x}|\Gamma(ix-1/4)|^2 \\ & \times\exp(-(4x)^{2/3}t/\tau_c - \pi x)dx, \qquad (32)\end{aligned}$$

where

$$\tau_n = \frac{\tau_c}{(4n-1)^{2/3}},$$

$$|\Gamma(ix-1/4)|^2 = |\Gamma(-1/4)|^2\prod_{n=0}^{\infty}\frac{(n-1/4)^2}{x^2+(n-1/4)^2}.$$

The right hand side of the integral equation (31) expresses the imposition of the boundary perturbation. It is related the the fact that the initial evolution of the reconnection (27) is dominated by the imposition function, $\psi_e(t/\tau_e)$. Since the kernel $G(t)$ expresses the response of the inner layer for the boundary perturbation, the time scale in the exponential function in $G(t)$ is the reconnection time scale $\tau_c \propto \tau_A^{2/3}\tau_R^{1/3}$. At $t = 0$ the integral part vanishes and $\psi_e(0) = 0$, thus the reconnected flux vanishes at the initial time, $\psi_1(0,0) = 0$, to satisfy the initial condition.

The integral equation for the inner layer reconnected flux $\psi_{inner}(0,t)$ is deduced in the same way as the equation for the reconnected flux $\psi_1(0,t)$.

6 Summary and discussion

We have corrected the previous boundary layer analysis of the forced reconnection due to the external boundary perturbation to be appropriate for the following points. One is the matching condition. With the exact matching condition, the effect of the inertia of the plasma in the inner layer is included correctly. The other is the imposition of the boundary perturbation. We have adopted the time dependent imposition of the boundary perturbation so that the outer region obeys the ideal MHD equilibrium equations. By correcting these points, we derived the new Laplace transformed reconnected flux with the exact solution of the linearized reduced magnetohydrodynamics equations for the inner layer equation.

Since the effect of the inertia is exactly included, the exact matching conditions lead to the two reconnected fluxes: the reconnected flux and the inner-layer reconnected flux. The former represents the global deformation of the magnetic field lines with the changing of the quasi-static equilibrium. The later represents the real reconnection at the neutral surface in the inner layer. It is shown that the characteristic time scale of the reconnection in the initial evolution is significantly different from the one in the previous works[2, 4, 5]; the time scale of these reconnected fluxes include the time scale of the imposition of the boundary perturbation. Therefore it appears that the initial evolution of the forced reconnection is strongly affected by the imposition time scale and could be faster than the evolution with the Sweet-Parker time scale.

The boundary perturbation induces the local current on the resonant surface. In the initial evolution the local current has the negative sign to suppress the growth of the magnetic islands. This suppression stems from the fact that the initial equilibrium is stable in the absence of the boundary perturbation. For the forced reconnection, the instability parameter Δ', that is related to the local current on the resonant surface, varies with time, while Δ' is often fixed for the usual tearing modes. By virtue of the exact asymptotic matching, we have $\Delta' = 0$ at $t = 0$ and it increases with the negative sign, while $\Delta' \to \infty$ at $t = 0$ in the previous works[2, 4, 5].

These results implies the modification of the previous estimation for the transition from the linear to the nonlinear stage.[4] The modification of the transition is expected to have a significant effect on the time scale of the islands growth and the decay of the local current on the resonant surface in the nonlinear evolution.

A new method is also proposed in terms of an integral equation for the time evolution of the reconnected flux. The subsequent evolution of the initial evolution will be obtained by use of the integral equation, in the following paper.

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